APPLICATION OF LDTM FOR SOLVING HEAT-LIKE AND WAVE-LIKE EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT

In this article, a coupling of Laplace transformation and Differential transform method is presented for solving heat-like and wave-like equations with variable coefficients. We demonstrate that the proposed method is very convenient for achieving the analytical solutions of 2D and 3D partial differential equations. The numerical computation shows the efficiency and simplicity of the method.

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KEYWORDS: *DTM, Laplace transformation, Heat equation, Wave equation.*

1. INTRODUCTION

The heat-like and wave-like equations can be found in a wide variety of engineering and scientific applications. In recent years, many analytical and advanced methods are developed for heat-like equations, wave-like equations and wave systems [1-4]. The Differential Transform Method (DTM) is one of them. DTM is an analytical approach based on Taylor series expansion was firstly applied in the engineering field by J.K. Zhou in 1986 for solving linear and nonlinear equation associated with electrical circuit analysis [5]. DTM has been successfully applied to solve different types of heat-like and wave-like equations [6]. In this paper, the coupling of differential transform method and Laplace transformation is applied to obtain exact solutions of heat-like and wave-like equations with variable coefficients. The Laplace-differential transform method (LDTM) is an approximate analytical technique

for solving partial differential equations introduced by Marwan Alquran *et* al. [7] and it has been successfully applied for solving different types of physical problems such as Cauchy reaction diffusion equations and diffusion equation by Kiranta *et* al. [8-9]. The suggested algorithm is tested on 2-dimensional and 3-dimensional heat-like and wave-like equations. To the best of our knowledge no such try has been made to combine LTM and DTM for solving 3-dimensional heat-like and wave-like equations. Three examples for heat-like equations and three examples of wave-like equations are solved to make clear the application of the transform and the numerical results are very encouraging.

Heat-Like Equation

We consider a heat-like equation with variable coefficients described by a three-dimensional initial value problem (IVP) of the form

$$u_t = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}, \quad 0 < x < a, 0 < y < b, 0 < z < c, t > 0,$$

with the initial conditions, (1.1)

$$u(x, y, z, 0) = \phi(x, y, z). \tag{1.2}$$

Wave-Like Equation

We consider a wave-like equation with variable coefficients described by a three-dimensional initial value problem (IVP) of the form

$$u_{tt} = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}, \quad 0 < x < a, 0 < y < b, 0 < z < c, t > 0, \tag{1.3}$$

with the initial conditions

$$u(x, y, z, 0) = \phi(x, y, z), \qquad u_t(x, y, z, 0) = \varphi(x, y, z). \tag{1.4}$$

2. N-Dimensional Differential Transformation

The differential transform of a function $u(x_1, x_2, ..., x_n, t)$ is defined as:

$$U_{k_1, k_2, ..., k_n}(t) = \frac{1}{k_1! k_2! ... k_n!} \left[\frac{\partial^{k_1 + k_2 + ... + k_n} u(x_1, x_2, ..., x_n, t)}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n}} \right]_{x_1 = 0, x_2 = 0, ..., x_n = 0} ; k \geq 0 \tag{2.1}$$

where $u(x_1, x_2, ..., x_n, t)$ is the original function and $U_{k_1, k_2, ..., k_n}(t)$ is the transformed function. The inverse differential transform of $U_{k_1, k_2, ..., k_n}(t)$ is defined as:

$$u(x_1, x_2, ..., x_n, t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} ... \sum_{k_n=0}^{\infty} U_{k_1, k_2, ..., k_n}(t) x_1^{k_1} x_2^{k^2} ... x_n^{k_n}, \tag{2.2}$$

In actual applications, the function $u(x_1, x_2, ..., x_n, t)$ is expressed by a finite series and equation (2.2) can be written as

$$u(x_1, x_2, ..., x_n, t) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} ... \sum_{k_n=0}^{m_n} U_{k_1, k_2, ..., k_n}(t) x_1^{k_1} x_2^{k^2} ... x_n^{k_n}. \tag{2.3}$$

The fundamental mathematical operations performed by n-Dimensional Differential Transform are listed in the following Table 1.

Original Function	Transformed Function
$u(x_1, x_2, ..., x_n, t) = f(x_1, x_2, ..., x_n, t) \pm g(x_1$	$U_{k_1, k_2, ..., k_n}(t) = F_{k_1, k_2, ..., k_n}(t) \pm G_{k_1, k_2, ..., k_n}(t$
$u(x_1, x_2, ..., x_n, t) = \alpha\, f(x_1, x_2, ..., x_n, t)$	$U_{k_1, k_2, ..., k_n}(t) = \alpha\, F_{k_1, k_2, ..., k_n}(t)$
$u(x_1, x_2, ..., x_n, t) = \dfrac{\partial^{r_1, +r_2 + ... + r_n} u(x_1, x_2, ..., x_n}{\partial x_1^{r_1} \partial x_2^{r_2} ... \partial x_n^{r_n}}$	$U_{k_1, k_2, ..., k_n}(t) = (k_1 + 1)...(k_2 + r_1)...(k_n + 1)$ $...(k_n + r_n) F_{k_1 + r_1, ..., k_n + r_n}(t)$
$u(x_1, x_2, ..., x_n, t) = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$	$U_{k_1, k_2, ..., k_n}(t) = \delta(k_1 - a_1, k_2 - a_2, ..., k_n - a_n$
$u(x_1, x_2, ..., x_n, t) = f(x_1, x_2, ..., x_n, t) g(x_1, x_2,$	$U_{k_1, k_2, ..., k_n}(t) = \sum_{r_1=0}^{k_1} \sum_{r_2=0}^{k_2} ... \sum_{r_n=0}^{k_n} F_{r_1, ..., r_{n-1}, k_n - r_n}(t)$ $G_{k_1 - r_1, k_2 - r_2, ..., r_n}(t)$

3. BASIC IDEA OF LDTM

To illustrate the basic idea of Laplace differential transform method [7], we consider the heat-like and wave-like equations.

3.1 SOLUTION OF THE HEAT-LIKE EQUATION BY LDTM

We consider a heat-like equation with variable coefficients described by a three-dimensional initial value problem (IVP) of the form

$$u_t = f(x, y, z) u_{xx} + g(x, y, z) u_{yy} + h(x, y, z) u_{zz}, \quad x \in R, t \in R^+, \tag{3.1}$$

with the initial conditions,

$$u(x, y, z, 0) = \phi(x, y, z) \tag{3.2}$$

and the spatial conditions

$$u(0, y, z, t) = \alpha_1(y, z, t), \quad u(x, 0, z, t) = \alpha_2(x, z, t), \; u(x, y, 0, t) = \alpha_3(x, y, t). \tag{3.3}$$

Taking the Laplace Transformation of equation (3.1), w.r.to‘t’, we get

$$s\, L[u(x, y, z, t)] - u(x, y, z, 0) = L\left[f(x, y, z) u_{xx} + g(x, y, z) u_{yy} + h(x, y, z) u_{zz}\right].$$

By using I.C. (3.2), we get

$$L[u(x, y, z, t)] = \frac{\phi(x, y, z)}{s} + \frac{1}{s} L\left[f(x, y, z) u_{xx} + g(x, y, z) u_{yy} + h(x, y, z) u_{zz}\right]. \tag{3.4}$$

Now, applying the DTM on the equation (3.4) with respect to 'x', 'y', 'z', we get

$$L\left[U_{k,h,m}(t)\right] = \frac{1}{s}\phi(k,h,m) + \frac{1}{s}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(r+2)(r+1)U_{r+2,s,m-l}(t)F_{k-r,h-s,l}(t)\right] +$$
$$\frac{1}{s}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(s+2)(s+1)U_{r,s+2,m-l}(t)G_{k-r,h-s,l}(t)\right] +$$
$$\frac{1}{s}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(m-l+2)(m-l+1)U_{r,s,m-l}(t)H_{k-r,h-s,l}(t)\right]. \tag{3.5}$$

Taking the inverse Laplace transformation of equation (3.5), we get

$$U_{k,h,m}(t) = \phi(k,h,m) + L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(r+2)(r+1)U_{r+2,s,m-l}(t)F_{k-r,h-s,l}(t)\right]\right] +$$
$$L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(s+2)(s+1)U_{r,s+2,m-l}(t)G_{k-r,h-s,l}(t)\right]\right] +$$
$$L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(m-l+2)(m-l+1)U_{r,s,m-l}(t)H_{k-r,h-s,l}(t)\right]\right]. \tag{3.6}$$

Now, applying the DTM on the given spatial condition (3.3), we get

$$U_{0,h,m}(t) = \alpha_1(y,z,t), U_{k,0,m}(t) = \alpha_2(x,z,t), U_{k,h,0}(t) = \alpha_3(x,y,t). \tag{3.7}$$

Now using the equation (3.7) in (3.6), the solution in the series form is given by

$$u(x,y,z,t) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}U_{k,h,m}(t)x^k y^h z^m .$$

3.2 SOLUTION OF THE WAVE-LIKE EQUATION BY LDTM

We consider a wave-like equation with variable coefficients described by a three-dimensional initial value problem (IVP) of the form

$$u_{tt} = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + h(x,y,z)u_{zz}, \quad x \in R, t \in R^+, \tag{3.8}$$

with the initial conditions,

$$u(x,y,z,0) = \phi(x,y,z), \quad u_t(x,y,z,0) = \varphi(x,y,z), \tag{3.9}$$

and the spatial conditions

$$u(0,y,z,t) = \alpha_1(y,z,t), \quad u(x,0,z,t) = \alpha_2(x,z,t), \ u(x,y,0,t) = \alpha_3(x,y,t) \tag{3.10}$$

and

$$u_x(0, y, z, t) = \beta_1(y, z, t), \quad u_y(x, 0, z, t) = \beta_2(x, z, t), \; u_z(x, y, 0, t) = \beta_3(x, y, t). \tag{3.11}$$

Taking the Laplace Transformation of equation (3.8), w.r.to't', we get

$$s^2 L[u(x, y, z, t)] - su(x, y, z, 0) - u_t(x, y, z, 0) = L[f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}].$$

By using I.C. (3.9), we get

$$L[u(x, y, z, t)] = \frac{1}{s}\phi(x, y, z) + \frac{1}{s^2}\varphi(x, y, z) + \frac{1}{s^2}L[f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}]. \tag{3.12}$$

Now, applying the DTM on the equation (3.12) with respect to 'x', 'y', 'z', we get

$$\begin{aligned} L[U_{k,h,m}(t)] &= \frac{1}{s}\phi(k, h, m) + \frac{1}{s^2}\varphi(k, h, m) + \\ &\frac{1}{s^2}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(r+2)(r+1)U_{r+2,s,m-l}(t)F_{k-r,h-s,l}(t)\right] + \\ &\frac{1}{s^2}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(s+2)(s+1)U_{r,s+2,m-l}(t)G_{k-r,h-s,l}(t)\right] + \\ &\frac{1}{s^2}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(m-l+2)(m-l+1)U_{r,s,m-l}(t)H_{k-r,h-s,l}(t)\right]. \end{aligned} \tag{3.13}$$

Taking the inverse Laplace transformation of equation (3.13), we get

$$\begin{aligned} U_{k,h,m}(t) &= \phi(k, h, m) + t\varphi(k, h, m) + \\ &L^{-1}\left[\frac{1}{s^2}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(r+2)(r+1)U_{r+2,s,m-l}(t)F_{k-r,h-s,l}(t)\right]\right] + \\ &L^{-1}\left[\frac{1}{s^2}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(s+2)(s+1)U_{r,s+2,m-l}(t)G_{k-r,h-s,l}(t)\right]\right] + \\ &L^{-1}\left[\frac{1}{s^2}L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(m-l+2)(m-l+1)U_{r,s,m-l}(t)H_{k-r,h-s,l}(t)\right]\right]. \end{aligned} \tag{3.14}$$

Now, applying the DTM on the given spatial condition (3.10) and (3.11), we get

$$U_{0,h,m}(t) = \alpha_1(y, z, t), U_{k,0,m}(t) = \alpha_2(x, z, t), U_{k,h,0}(t) = \alpha_3(x, y, t) \tag{3.15}$$

and

$$U_{1,h,m}(t) = \beta_1(y, z, t), U_{k,1,m}(t) = \beta_2(x, z, t), U_{k,h,1}(t) = \beta_3(x, y, t). \tag{3.16}$$

Now using the equation (3.15) and (3.16) in (3.14), the solution in the series form is given by

$$u(x, y, z, t)=\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}U_{k,h,m}(t)\,x^k y^h z^m \ .$$

4. NUMERICAL APPLICATIONS

4.1. HEAT-LIKE MODELS

In this section, three Heat-like models from each type will be tested by using the LDTM.
Example 1**.** Consider the one-dimensional heat-like model

$$\frac{\partial u}{\partial t}=\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad 0<x<1,\ t>0, \tag{4.1}$$

with the initial conditions,

$$u(x,0)=x^2, \tag{4.2}$$

In this technique, first we apply the Laplace transformation on equation (4.1) with respect to 't', therefore, we get

$$s\,L[u(x,t)]-u(x,0)=L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

By using initial conditions from equation (4.2), we get

$$L[u(x,t)]=\frac{x^2}{s}+\frac{1}{s}L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

Now, we applying the Inverse Laplace transformation w.r.t. 's' on both sides:

$$u(x,t)=x^2+L^{-1}\left[\frac{1}{s}L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right]\right].$$

The next step is applying the Differential transformation method with respect to space variable 'x', we get

$$U_k(t)=\delta(k-2,t)+L^{-1}\left[\frac{1}{2s}L\left[\sum_{r=0}^{k}(r+2)(r+1)U_{r+2}(t)\delta(k-r-2,t)\right]\right];$$

By straightforward iterative steps, we get the component $U_k(t), k\geq 0$ of the DTM can be obtained.

$$U_k(t) = \begin{cases} 1 & k = 1 \\ e^t & k = 2 \\ 0 & else \end{cases} \tag{4.3}$$

Finally, the closed form solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t) x^k = x^2 e^t.$$

which is the exact solution.

Example 2**.** Consider the two-dimensional heat-like model

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left[y^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} \right], \quad 0 < x,\, y < 1,\ t > 0, \tag{4.4}$$

with the initial conditions,

$$u(x, y, 0) = 2y^2. \tag{4.5}$$

In this technique, first we apply the Laplace transformation on equation (4.4) with respect to ‘t’, therefore, we get

$$s\, L[u(x, y, t)] - u(x, y, 0) = \frac{1}{2} L\left[y^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} \right].$$

By using initial conditions from equation (4.5), we get

$$L[u(x, y, t)] = \frac{2y^2}{s} + \frac{1}{2s} L\left[y^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} \right].$$

Now, we applying the Inverse Laplace transformation w.r.t. ‘s’ on both sides:

$$u(x, y, t) = 2y^2 + L^{-1}\left[\frac{1}{2s} L\left[y^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} \right]\right].$$

The next step is applying the Differential transformation method with respect to space variable ‘x’ and ‘y’, we get

$$U_{k,h}(t) = 2\delta(k, h-2, t) + L^{-1}\left[\frac{1}{2s} L\left[\sum_{r=0}^{k} \sum_{s=0}^{h} (r+2)(r+1) U_{r+2,\, h-s}(t) \delta(k-r, s-2, t) \right]\right] +$$
$$L^{-1}\left[\frac{1}{2s} L\left[\sum_{r=0}^{k} \sum_{s=0}^{h} (s+2)(s+1) U_{k-r,\, s+2}(t) \delta(r-2, h-s, t) \right]\right];$$

By straightforward iterative steps, we get the component $U_{k,h}(t), k \geq 0, h \geq 0$ of the DTM can be obtained.

$$U_{k,h}(t)=\begin{cases}2\sinh(t) & k=2,h=0\\ 2\cosh(t) & k=0,h=2\\ 0 & else\end{cases}$$

(4.6)

Finally, the closed form solution is given by

$$u(x, y, t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{k,h}(t)\,x^k y^h = 2x^2 \sinh(t)+2y^2 \cosh(t).$$

which is the exact solution.

Example 3. Consider the three-dimensional heat-like model

$$\frac{\partial u}{\partial t}=x^4y^4z^4+\frac{1}{36}\left[x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right],\quad 0<x,y,z<1,\ t>0,$$

(4.7)

with the initial conditions,

$$u(x, y, z, 0)=0.$$

(4.8)

In this technique, first we apply the Laplace transformation on equation (4.7) with respect to ‘t’, therefore, we get

$$sL[u(x, y, z, t)]-u(x, y, z,0)=\frac{1}{s}\left[x^4y^4z^4\right]+\frac{1}{36}L\left[x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right].$$

By using initial conditions from equation (4.8), we get

$$L[u(x, y, z, t)]=\frac{1}{s^2}\left[x^4y^4z^4\right]+\frac{1}{36\,s}L\left[x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right].$$

Now, we applying the Inverse Laplace transformation w.r.t. ‘s’ on both sides:

$$u(x, y, z, t)=t\left[x^4y^4z^4\right]+L^{-1}\left[\frac{1}{36\,s}L\left[x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right]\right].$$

The next step is applying the Differential transformation method with respect to space variable ‘x’, ‘y’ and ‘z’, we get

$$U_{k,h,m}(t) = t\delta(k-4, h-4, m-4, t) +$$
$$L^{-1}\left[\frac{1}{36s} L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(r+2)(r+1)U_{r+2,s,m-l}(t)\delta(k-r-2, h-s, l, t)\right]\right] +$$
$$L^{-1}\left[\frac{1}{36s} L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(s+2)(s+1)U_{r,s+2,m-l}(t)\delta(k-r, h-s-2, l, t)\right]\right] +$$
$$L^{-1}\left[\frac{1}{36s} L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(m-l+2)(m-l+1)U_{r,s,m-l+2}(t)\delta(k-r, h-s, l-2, t)\right]\right];$$

By straightforward iterative steps, we get the component $U_{k,h,m}(t), k \geq 0, h \geq 0$ and $m \geq 0$ of the DTM can be obtained.

$$U_{k,h,m}(t) = \begin{cases} e^t - 1 & k = 4, h = 4, m = 4 \\ 0 & else \end{cases} \qquad (4.9)$$

Finally, the closed form solution is given by

$$u(x, y, z, t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty} U_{k,h,m}(t)\, x^k y^h z^m = x^4 y^4 z^4 (e^t - 1).$$

which is the exact solution.

4.2. WAVE-LIKE MODELS

In this section, we illustrate our analysis by examining the following three Wave-like equations.

Example 4. Consider the one-dimensional wave-like model

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,\ t > 0, \qquad (4.10)$$

with the initial conditions,

$$u(x,0) = x, \quad u_t(x,0) = x^2, \qquad (4.11)$$

In this technique, first we apply the Laplace transformation on equation (4.10) with respect to ‘t’, therefore, we get

$$s^2 L[u(x,t)] - su(x,0) - u_t(x,0) = L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

By using initial conditions from equation (4.11), we get

$$L[u(x,t)] = \frac{x}{s} + \frac{x^2}{s^2} + \frac{1}{s^2} L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

Now, we applying the Inverse Laplace transformation w.r.t. ‘s’ on both sides:

$$u(x,t) = x + x^2 t + L^{-1}\left[\frac{1}{s^2} L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right]\right].$$

The next step is applying the Differential transformation method with respect to space variable ‘x’, we get

$$U_k(t) = \delta(k-1,t) + t\delta(k-2,t) + L^{-1}\left[\frac{1}{2s^2} L\left[\sum_{r=0}^{k}(r+2)(r+1)U_{r+2}(t)\delta(k-r-2,t)\right]\right];$$

By straightforward iterative steps, we get the component $U_k(t), k \geq 0$ of the DTM can be obtained.

$$U_k(t) = \begin{cases} 1 & k = 1 \\ \sinh(t) & k = 2 \\ 0 & else \end{cases} \tag{4.12}$$

Finally, the closed form solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)\, x^k = x + x^2 \sinh(t).$$

which is the exact solution.

Example 5. Consider the two-dimensional wave-like model

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{12}\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}\right], \quad 0 < x,\, y < 1,\; t > 0, \tag{4.13}$$

with the initial conditions,

$$u(x, y, 0) = x^4, \quad u_t(x, y, 0) = y^4. \tag{4.14}$$

In this technique, first we apply the Laplace transformation on equation (4.13) with respect to ‘t’, therefore, we get

$$s^2 L[u(x, y, t)] - su(x, y,0) - u_t(x, y,0) = \frac{1}{12} L\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}\right].$$

By using initial conditions from equation (4.14), we get

$$L[u(x, y, t)] = \frac{x^4}{s} + \frac{y^4}{s^2} + \frac{1}{12s^2} L\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}\right].$$

Now, we applying the Inverse Laplace transformation w.r.t. 's' on both sides:

$$u(x, y, t) = x^4 + ty^4 + L^{-1}\left[\frac{1}{12s^2} L\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}\right]\right].$$

The next step is applying the Differential transformation method with respect to space variable 'x' and 'y', we get

$$U_{k,h}(t) = \delta(k-4, h, t) + t\delta(k, h-4, t) +$$
$$L^{-1}\left[\frac{1}{12s^2} L\left[\sum_{r=0}^{k} \sum_{s=0}^{h} (r+2)(r+1) U_{r+2,\, h-s}(t) \delta(k-r-2, s, t)\right]\right] +$$
$$L^{-1}\left[\frac{1}{12s^2} L\left[\sum_{r=0}^{k} \sum_{s=0}^{h} (s+2)(s+1) U_{k-r,\, s+2}(t) \delta(r, h-s-2, t)\right]\right];$$

By straightforward iterative steps, we get the component $U_{k,h}(t), k \geq 0, h \geq 0$ of the DTM can be obtained.

$$U_{k,h}(t) = \begin{cases} \cosh(t) & k = 4, h = 0 \\ \sinh(t) & k = 0, h = 4 \\ 0 & else \end{cases} \tag{4.15}$$

Finally, the closed form solution is given by

$$u(x, y, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{k,h}(t) x^k y^h = x^4 \cosh(t) + y^4 \sinh(t).$$

which is the exact solution.

Example 6. Consider the three-dimensional wave-like model

$$\frac{\partial^2 u}{\partial t^2} = x^2 + y^2 + z^2 + \frac{1}{2}\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2}\right], \quad 0 < x, y, z < 1, \; t > 0, \tag{4.16}$$

with the initial conditions,

$$u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = x^2 + y^2 - z^2. \tag{4.17}$$

In this technique, first we apply the Laplace transformation on equation (4.16) with respect to 't', therefore, we get

$$s^2 L[u(x, y, z, t)] - su(x, y, z, 0) - u_t(x, y, z, 0) = \frac{1}{s}\left[x^2 + y^2 + z^2\right] + \frac{1}{2} L\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2}\right].$$

By using initial conditions from equation (4.17), we get

$$L[u(x, y, z, t)] = \frac{1}{s^2}\left[x^2 + y^2 - z^2\right] + \frac{1}{s^3}\left[x^2 + y^2 + z^2\right] + \frac{1}{2s^2} L\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2}\right].$$

Now, we applying the Inverse Laplace transformation w.r.t. 's' on both sides:

$$u(x, y, z, t) = t\left[x^2 + y^2 - z^2\right] + \frac{t^2}{2}\left[x^2 + y^2 + z^2\right] + L^{-1}\left[\frac{1}{2s^2} L\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2}\right]\right].$$

The next step is applying the Differential transformation method with respect to space variable 'x', 'y' and 'z', we get

$$\begin{aligned}
U_{k,h,m}(t) &= \left[t + \frac{t^2}{2}\right]\delta(k-2, h, m, t) + \left[t + \frac{t^2}{2}\right]\delta(k, h-2, m, t) - \left[t - \frac{t^2}{2}\right]\delta(k, h, m-2, t) \\
& L^{-1}\left[\frac{1}{2s^2} L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(r+2)(r+1)U_{r+2,s,m-l}(t)\delta(k-r-2, h-s, l, t)\right]\right] + \\
& L^{-1}\left[\frac{1}{2s^2} L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(s+2)(s+1)U_{r,s+2,m-l}(t)\delta(k-r, h-s-2, l, t)\right]\right] + \\
& L^{-1}\left[\frac{1}{2s^2} L\left[\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{l=0}^{m}(m-l+2)(m-l+1)U_{r,s,m-l+2}(t)\delta(k-r, h-s, l-2, t)\right]\right];
\end{aligned}$$

By straightforward iterative steps, we get the component $U_{k,h,m}(t), k \geq 0, h \geq 0$ and $m \geq 0$ of the DTM can be obtained.

$$U_{k,h,m}(t) = \begin{cases} e^{t} - 1 & k = 2, h = 0, m = 0 \\ e^{t} - 1 & k = 0, h = 2, m = 0 \\ e^{-t} - 1 & k = 0, h = 0, m = 2 \\ 0 & otherwise \end{cases} \tag{4.18}$$

Finally, the closed form solution is given by

$$u(x, y, z, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} U_{k,h,m}(t) x^k y^h z^m = x^2 (e^t - 1) + y^2 (e^t - 1) + z^2 (e^{-t} - 1).$$

which is the exact solution.

5. CONCLUSION

In this study, we apply Differential Transform Method (DTM) coupled with Laplace Transformation is presented for solving heat-like and wave-like equations with variable coefficients which arise very frequently in physical problems related to applied sciences and engineering [2], [6]. We demonstrate that the proposed method is very convenient for achieving the analytical solutions of 2-D and 3-D partial differential equations. It is observed that the proposed technique is suitable for such type of problems, it gives rapidly converging series solutions and gives excellent accuracy for finding and is very user-friendly. Computational work and subsequent results are fully supportive of the reliability, simplicity, efficiency and accuracy of the suggested scheme.

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